Chapter 2

Mathematical Methods

■ 2.1. Motivation

Actually enough already has been written about the solving methods of linear equations (e.g. [Metz1996], section 3.5, p. 23-26). Nevertheless in this chapter the application of Fourier and Laplace techniques for generating propagators and Green's functions is indicated shortly and extended.

The main reason to this chapter is the elucidation of the applicability of Dirac's *delta function* as an analytical function being consistent due to function theory. Further more is to be shown, how by use of it also more complicated linear differential equations (also fractional) can be solved eventually, especially concerning the calculation of momenta.

The results being presented here give a fundamental theoretical extension, without which a consistent realization of the software package *FractionalCalculus* would not have been possible. The following elaboration presents the basics in such a way, that knowledge of the software package indeed is useful, but not necessary to be able to follow the content.

■ 2.2. Dirac's Delta Function

■ 2.2.1. Singular Integrals

If a definite integral diverges for a certain choice of the parameters, then thoroughly it makes sense to present the whole solution in a possibly united form.

For the presentation of these potential singular integration results also among others Dirac's delta function is suitable. Nevertheless yet today some mathematicians meet it with suspicion, because the transition into the singularity is done non-continually.

■ 2.2.2. Fourier Transformation of Number One

■ 2.2.2.1. Properties of Fourier Transformation

The Fourier transformation of a constant can be reduced to the one of number one. The Fourier integral itself reads concerning this elaboration and at the *Mathematica* version 3.0 (There are also deviations from this in other elaborations concerning the multiplicator and the sign, e.g. at *Mathematica* 4.0):

$$\mathcal{F}_x^k[f[x]] := \int_{-\infty}^{\infty} \mathrm{Exp}[+i\,k\,x]\; f[x]\,dx\,. \tag{2.1}$$

With symmetrical functions $f[-x] == f[x]$ the Fourier transformation can convert to the cosine transformation:

$$\overset{c\,k}{\mathcal{F}}_x[f[x]] := \int_0^{\infty} \mathrm{Cos}[k\,x]\; f[x]\,dx\,. \tag{2.2}$$

Further properties of the Fourier transformations are listed in appendix B of this elaboration.

■ 2.2.2.2. Application of the Mean Value Theorem

The Fourier transformation of number one leads from equation (2.1) to the double of equation (2.2), where now a parametric integral is to be solved:

$$\mathcal{F}_x^k[1] = 2\,\overset{c\,k}{\mathcal{F}}_x[1] = 2\int_0^{\infty} \mathrm{Cos}[k\,x]\,dx = \begin{cases} \infty & k = 0\,, \\ 0 & k \neq 0\,. \end{cases} \tag{2.3}$$

The calculated results can be understood well via the mean value theorem ([Rot1954], II. §13.9, p. 86-87), because for $k \neq 0$ the mean value of infinitely many periods of the cosine function is determined. The integral of finitely many cosine periods anyway gives zero.

Other accesses to the integral (2.3) result indeterminated terms. The formulation of an unequivocal result indeed is possible, but not accepted everywhere.

■ 2.2.2.3. Definition of the Delta Function due to Dirac

P. A. M. Dirac now gives a function, that has got the properties of equation (2.3) and is able to give back number one by inverse Fourier transformation. This delta function is defined by the following three assumptions ([Dir1927], p. 625-626):

$$\delta[x] = 0 \qquad x \neq 0\,, \tag{2.4}$$

$$\int_{-\infty}^{\infty} dx\, \delta[x] = 1\,, \tag{2.5}$$

$$\delta[x] = \delta[\,|\,x\,|]\,. \tag{2.6}$$

In the original elaboration of Dirac there is a misprint with the formulation of definition (2.5), which unfortunately intensifies the possibilities of critizism to his publication.

A consequence being consistent due to function theory results from the definitions (2.4) to (2.6) in a complex phase φ of the integral depending on the direction of the integration path, which is presented here already in a possibly general useful presentation:

$$\int_{0}^{\mathrm{Exp}[i\,\varphi]} \delta[x]\, dx = \left(\frac{\mathrm{Exp}[i\,\varphi]}{2}\right). \tag{2.7}$$

It should be mentioned, that already Dirac in his elaboration discusses the delta function having a complex argument ([Dir1927], eq. (1), p. 626); this result as yet is fundamental:

$$\int_{-\infty}^{\infty} f[x]\, \delta[a - x]\, dx = f[a]\,. \tag{2.8}$$

■ 2.2.2.4. Determination of the Multiplicator of the Fourier Transformation

Because the delta function plays an important role with the solution theory of linear equations, a sensible convention of mathematics consists in setting the Fourier transform of the delta function to number one. By the above given definition of the Fourier transformation (2.1) in connection to equation (2.8) results the following:

$$\mathcal{F}_x^k[\delta[x]] = 2\, {}^{c}\mathcal{F}_x^{k}[\delta[x]] = 2 \int_{0}^{\infty} \mathrm{Cos}[k\,x]\, \delta[x]\, dx = 1\,. \tag{2.9}$$

Therefore the Fourier transform of number one is proportional to Dirac's delta function. The multiplicator of proportionality results from the inverse Fourier transformation by use of the self-inversity of the cosine transformation and equation (2.7):

$$(\mathcal{F}^{-1})_k^x[2\pi\,\delta[k]] = \frac{1}{\pi}\,{}^{c}\mathcal{F}_k^x[2\pi\,\delta[k]] = \frac{2\pi}{\pi}\int_0^\infty \mathrm{Cos}[k\,x]\,\delta[k]\,dk = 1\,. \tag{2.10}$$

Thus the delta function gives clearness, if the multiplicators of the Fourier transformation are determined according to definition (2.1). This clearness at the latest is necessary with the use of computer algebra.

■ 2.2.3. Mellin Transformation of Number One

■ 2.2.3.1. Result from Fourier Transformation

Already Mellin directs to a close relationship between Fourier and Mellin transformation ([Mel1910], §8, p. 324). An explicit carrying-out of the corresponding transformations results by use of the substitutions $\boldsymbol{t \to -\mathrm{Log}[x]}$ and $\boldsymbol{\zeta \to i\,z}$:

$$\begin{aligned} f[x] = f[e^{-t}] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\zeta t}\,d\zeta \int_{-\infty}^{\infty} f[e^{-t}]\,e^{i\zeta t}\,dt = \\ &= \frac{-i}{2\pi i}\int_{-\infty}^{\infty} x^{i\zeta}\,d\zeta \int_0^\infty f[x]\,x^{-i\zeta-1}\,dx = \\ &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} x^{-z}\,dz \int_0^\infty f[x]\,x^{z-1}\,dx\,. \end{aligned} \tag{2.11}$$

The provisional results of these transformations (2.11) result by the residue (2.10) and definition (2.6) the Mellin transform of number one, at which calculation the author was assisted kindly by Professor Dr. W. Wonneberger (Ulm):

$$\int_0^\infty x^{z-1}\,dx = 2\pi\,\delta[\zeta] = 2\pi\,\delta[i\,z] = 2\pi\,\delta[z]\,. \tag{2.12}$$

■ 2.2.3.2. Mellin Residue of the Delta Function

The inverse Mellin transform of the result (2.12) yields by the integral (2.7) according to the equations (2.11):

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\delta[z]\,x^{-z}\,dz=\left(\frac{1}{2\pi}\right). \tag{2.13}$$

Therefore in the meaning of Mellin transformation even the residue of the delta function is unequivocal in spite of the direction dependency of the integral (2.7). The calculation of this *Mellin residue* can fail by the formulae, that otherwise are valid for continuously attainable singularities.

■ 2.2.3.3. Independent Confirmation

For not to establish inconsistencies, there is great value in getting a new result by several derivations. Therefore the Mellin transform of number one (2.12) now is to be confirmed on an independent way. The integral itself ranks as being divergent for all real $\boldsymbol{z}$.

For $\boldsymbol{z \neq 1}$ the Mellin transform of number one yields within the integration intervals $\boldsymbol{0 \leq x \leq 1}$ and $\boldsymbol{1 \leq x \leq \infty}$ both of the following functions, which can be continued analytically on their part:

$$\int_0^\infty x^{z-1}\,dx=\int_0^1 x^{z-1}\,dx+\int_1^\infty x^{z-1}\,dx=\frac{1}{z}-\frac{1}{z}, \tag{2.14}$$

$$\text{Re}[z]>0 \text{ and } \text{Re}[z]<0 .$$

The word "**and**" in the result (2.14) confuses again and again, because indeed the nearby fallacy is nourished, that the real part of $\boldsymbol{z}$ would be obliged to be positive and negative *simultaneously*, but indeed it elucidates, that for the present there are two different validity intervals to be distinguished *simultaneously*, before each analytical continuation of the partial results takes place. The given integral already caused difficulties to the author concerning his diploma thesis ([Süd1997], sections 3.1.4.3 and 6.3.3, p. 42-44 and 93-94), which are solved now even for computer algebra.

For $\boldsymbol{z == 0}$ the Mellin transform of number one yields logarithmic singularities, which cannot be removed by an analytical continuation of convergent partial integrals:

$$\int_0^\infty \frac{dx}{x}=\text{Log}[x]\,\big|_0^\infty=\infty+\infty=\infty . \tag{2.15}$$

Since Mellin transformation constantly yields results without poles according to the established function and integral theory, always an analytical continuation within Mellin space into the singularities of the result is necessary, that inverse Mellin transform can take place via the residual theorem by use of these singularities (cmp. [Mel1910], validity intervals of eq. (40), p. 316).

When implementing Mellin transformation to computer algebra, the author very soon realized this property, because the singularities of the residual calculation always were located beyond the calculated validity interval of the Mellin transform.

As a retaining rule generally results, that the *validity intervals of Mellin space* indeed are correct for integral convergence, but *are to be ignored* otherwise! Maybe especially this property of Mellin transformation keeps many mathematicians from getting down to the possibilities of this transformation.

■ 2.2.3.4. Consequences from the Theorem by Mellin

The theorem by Mellin calls a pair of function and Mellin transform to be reciprocal functions and tells ([Mel1910], S. 323):

"One of two reciprocal functions can be identical
to null just if also the other one is equal to zero only."

In application to the just now by use of all calculation rules successfully gotten results (2.12) and (2.13) the theorem by Mellin is the *proof of existence to Dirac's delta function*.

The author thinks the theorem by Mellin to be one of the most difficult to understand and most unknown theorems of analytical mathematics. For its understanding an additional lecture *function theory III* would be necessary.

■ 2.2.4. The Momenta of the Delta Function

■ 2.2.4.1. Connection between Mellin Transformation and Momenta

The momentum integrals of order m of a distribution function $f[x]$ are defined the following ([BrS1987], section 5.1.3, p. 667 top):

$$\overline{x^m} = <(x^m)> = <(x^m),\ f[x]> := \int_{-\infty}^{\infty} x^m\ f[x]\, dx\,. \tag{2.16}$$

The Mellin transformation is given by the following integral ([Mel1910], §8, eq. (51), p. 319):

$$\mathcal{M}_x^z[f[x]] := \int_0^{\infty} x^{z-1}\ f[x]\, dx\,. \tag{2.17}$$

Symmetrical momenta are calculated with the absolute values $|x|^m$ instead of x^m, where the integration bounds presuppose a symmetrical distribution function $f[x] = f[|x|]$. The following relation between symmetrical momenta $<|x|^m>$ and Mellin transformation of a function results:

$$\begin{aligned} &< |x|^m > = < |x|^m,\ f[|x|] > = \\ &= 2\,\mathcal{M}_x^{m+1}[f[x]] = 2\int_0^{\infty} x^{m+1-1}\ f[x]\, dx\,. \end{aligned} \tag{2.18}$$

■ 2.2.4.2. Calculation of the Delta Momenta

The symmetrical momenta of the delta function yield with the equations (2.18), (2.7) and (2.6):

$$< |x|^m,\ \delta[x] > = \begin{cases} 1 & m = 0\,, \\ 0 & m > 0\,. \end{cases} \tag{2.19}$$

More generally formulated the Mellin transformation (2.17) of the delta function according to relation (2.7) yields a quite peculiar function:

$$\mathcal{M}_x^z[\delta[x]] = \left(\frac{0^{z-1}}{2}\right). \tag{2.20}$$

Function (2.20) is quite peculiar, because the inverse Mellin transformation of 0^z again leads back to the singulary delta function.

■ 2.2.4.3. Possibilities of Enlarging Function Theory

In the face of results like (2.20) also an experienced mathematician reaches his limits. The formal going on of calculation with the formally gotten formulae always yields sensible results, where even a computer does not find inconsistencies of this possibility.

The results (2.19) and (2.20) with (2.18) suggest the conclusion to define the function values of $\mathbf{0}^{\boldsymbol{z}}$ for $\boldsymbol{z} == \mathbf{0}$ non-continually to be number one. A direct access via limit calculation of the kind $\boldsymbol{z} \to \mathbf{0}$ then naturally is not possible, but moreover a motivation via an analogy to the Hausdorff dimension ([Non1996], eq. (2.4), p. 37) of parametric limits:

$$0^z := \lim_{\delta \to 0} \delta^z \,. \tag{2.21}$$

The peculiarity of the $\mathbf{0}^{\boldsymbol{z}}$ function is, that the absolute value of the function is unequivocal for all $\boldsymbol{z}$, while the complex phase especially for pure imaginary $\boldsymbol{z} \neq \mathbf{0}$ keeps ready a lot of surprises.

Therefore when estimating integral convergence, result expressions of the form $\boldsymbol{x}^{\boldsymbol{z}}$ give a clear answer for $\mathbf{Re}[x] > \mathbf{0}$ and eventually still for $\boldsymbol{x} == \mathbf{0}$ and $\mathbf{Re}[z] > \mathbf{0}$. Already the case $\boldsymbol{x} == \mathbf{0}$ and $\boldsymbol{z} == \mathbf{0}$ causes discussions, which within each of the corresponding context of the result can be finished, yet.

The use of definition (2.21) to be an analytical function of $\boldsymbol{z}$ seems to be as non-usual as the discussion of $\sqrt{-1}$ five hundred years ago. To work off unsettled questions of this kind may be the aim of later elaborations. The analytical relevance of the $\mathbf{0}^{\boldsymbol{z}}$ function is yet pointed out by the momenta of the delta function (2.19).

■ 2.3. Laplace and Fourier Transformation

■ 2.3.1. Known Relations

■ 2.3.1.1. Plan of Action

With Laplace transformation ([BrS1987], section 4.4.3.1, p. 634) of fractional diffusion equations the differentiation theorem and the convolution theorem are sufficient to change the integrated form of the time fractional differential equation (1.26) into an ordinary differential equation of the spacial coordinate x.

The integration constants, that are mentioned in Oldham/Spanier ([OS1974], section 8.1, p. 133-136) in connection to fractional derivatives, own no physical relevance, which results by the discussion of fractional initial value problems (references see chapter 1.3.2 of this elaboration). At the software package *FractionalCalculus* they optionally can be switched on via the option *OldhamSpanierConstants→True* during carrying-out of Laplace transformation.

The subsequent Fourier transformation of these ordinary differential equation succeeds, if all initial value problems for the first are set to Dirac's delta function. In the easiest case (also with fractional diffusion equations!) an algebraic equation is gotten, whose inverse Fourier and inverse Laplace transformation lead to an unequivocal fundamental system of propagators, which in the context of this elaboration is called the *optimized fundamental system*.

The total solution results by carrying-out each of a Fourier convolution of the initial value problems and the corresponding fundamental propagators.

Other solution techniques of linear differential equations e.g. lead to *Frobenius' normal form* ([HT1956], eq. (11c), p. 188) of a fundamental system.

■ 2.3.1.2. Idea of Propagator

The propagator is part of the optimized fundamental system of a partial linear differential equation. The corresponding initial value problem always is Dirac's delta function at time $t \to 0$.

With the solution strategy being discussed here the time coordinate is to be handled differently from the spacial coordinates, because the initial value problem describes the spatial distribution $f[x, t \to 0]$ and not—also being mathematically possible—the time development of dynamics at a very certain location $f[x \to 0, t]$.

The Fourier transformation of a differential equation can be understood via partial integration, if the so-called *natural boundary conditions*, namely the disappearance of the function and its derivatives at infinity, are fulfilled. With propagators this demand because of the start at $t == 0$ by a delta function is to be discussed as fulfilled.

■ 2.3.1.3. Idea of Green's Function

In opposite to a propagator a Green's function describes the standardized solution of an inhomogenous differential equation. Due to Hort/Thoma ([HT1956], §107-108, p. 178-182) hereby always one additional integration is to be done in comparison to homogeneous equations.

To get the basic function, which is used for an integration with the general inhomogenity or steering size $s[x, t]$, again Dirac's delta function, this time being time dependent, is used.

The gotten Green's function in the easiest case is subjected to a time Laplace convolution and a spatial Fourier convolution to get the general inhomogeneous solution of the original equation.

■ 2.3.2. Prolongations

■ 2.3.2.1. Laplace Transformation of the Delta Function

Due to equation (2.7) Dirac's delta function has got an unequivocal Laplace transform:

$$\mathcal{L}_t^p[\delta[t]] = \int_0^\infty e^{-p\,t}\,\delta[t]\,dt = \left(\frac{1}{2}\right). \tag{2.22}$$

A lot of publications deviate from this result. Exemplary the references may be given only, that occur at *Mathematica* 3.0 in a program comment to the package *Calculus`DiracDelta`*: Hoskins [Hos1979] or Antosik, Mikusinski and Sikorski [AMS1973] yield the result number one.

■ 2.3.2.2. Consequences for Computer Algebra

A mathematician calculating with pencil and paper is able to hold out a situative context of the delta function consistently to the end. In opposite a computer algebra system must lead to calculation errors with this kind of specialities. By this reason the demands for mathematics to be formulated consistently are principally heavier by use of computer algebra than by the traditional calculation with pencil and paper.

Because of the situative context with the Laplace transformation of the delta function actually no mature computer algebra system exists, where the delta function would be implemented consistently or uncontradicted. With *Mathematica* yet it has been possible to reach a consistent implementation of the delta function in the context of this elaboration. Now the result is not called *DiracDelta*, what would describe the historical facts better, but *SymmetricalDelta*, because this function name had not been used, yet.

In this context mainly an "open" computer algebra system turns out to be useful, because the user of the program in the case of need must practise himself the necessary corrections of the system. The software package *Mathematica* is "half open", what means, that eventual changes of functions are to be implemented by an own name.

■ 2.3.2.3. Laplace Convolution with Delta Function

The Laplace convolution with a delta function according to equation (2.7) yields half of the convoluted function:

$$\int_0^t \delta[\tau]\, f[t-\tau]\, d\tau = \int_0^t \delta[t-\tau]\, f[\tau]\, d\tau = \left(\frac{f[t]}{2}\right). \tag{2.23}$$

Certainly it is not very practical to keep always ones mind to this factor $\frac{1}{2}$ when comparing with literature. Moreover it is suitable to exchange the function $\mathbf{2\,\delta[t]}$ instead of $\boldsymbol{\delta[t]}$ with the inhomogenity of the equation when applying Laplace transformation. The Laplace transform of this then is number one, and Green's function results to be that function, whose Laplace convolution with the steering size yields the general solution.

■ 2.3.2.4. Space and Time Dependent Steering Functions

With space and time dependent steering functions $\boldsymbol{s[x,\ y,\ z,\ t]}$ the independency of the coordinates is used thus leading to the following more general inhomogenity to get Green's function:

$$s[x,\ y,\ z,\ t] \to 2\,\delta[x]\,\delta[y]\,\delta[z]\,\delta[t]\,. \tag{2.24}$$

On equations with spatial independent analytical coefficients, e.g. the fractional diffusion equation (1.26), Green's function is contained within the fundamental system of the propagators. This results from the Laplace transformation of a corresponding example equation for $\boldsymbol{\beta \in \mathbb{N}}$ **:**

$$\begin{gathered}\mathcal{L}_t^p\Big[\frac{\partial^\beta f[x,\ t]}{\partial t^\beta} - \frac{\partial^2 f[x,\ t]}{\partial x^2} = 2\,\delta[x]\,\delta[t]\Big] \\ \Longleftrightarrow \quad p^\beta\,\mathcal{L}_t^p[f[x,\ t]] - \sum_{n=0}^{\beta-1}\delta[x]\,p^{\beta-1-n} - \frac{\partial^2 \mathcal{L}_t^p[f[x,\ t]]}{\partial x^2} = \delta[x]\,.\end{gathered} \tag{2.25}$$

The total identity of one of the propagators to Green's function thus is shown for $\boldsymbol{n == \beta - 1}$. More difficult inhomogeneous equations eventually can be handled via the homogeneous solution (there are more solution techniques for this!).

■ 2.3.2.5. Space Dependent Analytical Coefficients

If the analytical coefficients of an equation are space dependent, a special handling is necessary to calculate the general solution. For this the integral of a Fourier convolution is viewed for the first:

$$f[x] * g[x] = \int_{-\infty}^{\infty} f[x-a]\, g[a]\, da = \int_{-\infty}^{\infty} f[a]\, g[x-a]\, da\,. \tag{2.26}$$

If now the propagator is build up, then at starting time $\boldsymbol{t \to 0}$ a normal delta convolution according to equation (2.8) is available. However at later time now no more a Fourier convolution is to be calculated, but an integral, which summarizes the spacial dependent propagators having the starting distribution as weight.

Certainly spacial dependent propagators are yielded only, if also accordingly $\frac{\boldsymbol{\delta[x-a]+\delta[x+a]}}{\boldsymbol{2}}$ is set instead of $\boldsymbol{\delta[x]}$ to be a symmetrical initial value problem. The same handling also is to be noted at the building of Green's function. In the Fourier convolution (2.26) both the propagators and also Green's function depend on the integration distance $\boldsymbol{a}$ only. With asymmetric equations also an asymmetric initial value problem of the form $\boldsymbol{\delta[x-a]}$ is to be started with to find the propagators.

These connection are of value to solve corresponding Fokker Planck equations (chapter 1.2.2.3 and 2.4.1.1 of this elaboration). Here they shall not be discussed further on, because they would exceed the setting of this elaboration.

■ 2.3.2.6. Fourier Transformation of the Riesz Operator

The Riesz operator is presented in Samko et al. ([SKM1993], eqs. (12.1)-(12.4), p. 214) in such a way, that all symmetric and asymmetric variants of the Riesz operators can be build up therewith.

When discussing diffusion without shifting of the focal point the following definition is sufficient, which also can be understood as Fourier convolution with the power of the absolute value function, and thus leads to the Mellin transformation of the cosine function ([Obe1974], formulae I.1.2 and I.5.2, p. 11 and 42; [EMOT1953], eqs. 1.2(7) and 1.2(15), p. 3 and 5):

$$\begin{aligned} \mathcal{F}_x^k[-|x|^{-\mu-1}] &= -\int_{-\infty}^{\infty} e^{ikx}\,(|x|)^{-\mu-1}\,dx = \\ &= -2\int_0^{\infty} \mathrm{Cos}[|k|\;x]\,x^{-\mu-1}\,dx = -\mathcal{M}_x^{-\mu}[\mathrm{Cos}[|k|\;x]] = \\ &= -(|k|)^{\mu}\,\Gamma[-\mu]\,\mathrm{Cos}\Big[-\frac{\mu\pi}{2}\Big] = -\frac{2^{-\mu}\sqrt{\pi}\,\Gamma[-\frac{\mu}{2}]}{\Gamma\big[\frac{\mu+1}{2}\big]}\,(|k|)^{\mu}\,. \end{aligned} \tag{2.27}$$

The Riesz operator to be found here is to be build up in such a way, that for $\boldsymbol{\mu \to 2}$ results the Fourier transform of the second derivative, namely $\boldsymbol{-k^2}$ as a multiplicator.

This aim is reached by the following Fourier convolution, where connects to the Riesz operator an integral order $\boldsymbol{-\mu}$ and $\boldsymbol{\mu}$ here shall present the fractional differentiation order:

$$\begin{aligned} -\mathcal{R}_x^{-\mu}[f[x]] &= -\frac{2^{\mu}\,\Gamma\big[\frac{\mu+1}{2}\big]}{\sqrt{\pi}\,\Gamma[-\frac{\mu}{2}]}\int_{-\infty}^{\infty}\frac{f[y]}{|x-y|^{\mu+1}}\,dy = \\ &= (\mathcal{F}^{-1})_k^x\Big[-\,|k\,(|^{\mu}\;\mathcal{F})_x^k[f[|k|]]\Big]\,. \end{aligned} \tag{2.28}$$

The formulation of this operator leads back to the elaboration of V. Seshadri and B. J. West [SWe1982]. The Fourier transformation of the Riesz operator being possible for driftless diffusion is given by equation (2.28).

■ 2.4. Mellin Transformation

■ 2.4.1. Difference Equations

■ 2.4.1.1. Generating Linear Difference Equations

Laplace transformation of a time fractional diffusion equation also leads for space dependent diffusion parameters—e.g. dependent of power terms due to the notation of Risken ([Ris1984], chap. 1.2.1, p. 4-5)—to an ordinary differential equation of the spatial coordinate, whose inhomogeneous solution is asked for—in the case of the given example to the basic equations of the Bessel functions.

Mellin transformation of an ordinary homogeneous differential equation leads to the corresponding difference equation ([Mes1959], chap. X.1, p. 133-134). If relation (2.25) is established at least for special initial value problems (e.g. for $\delta[x - 0]$), then the solution of the homogeneous difference equation of Mellin space is enough to solve the originally inhomogeneous differential equation.

■ 2.4.1.2. Solution to Linear Difference Equations

There are solution techniques (see [Mes1959]) of homogeneous and inhomogeneous linear difference equations. Sometimes it is possible to guess one of the homogeneous solutions of a difference equation, especially if the solving function of the original equation is a Fox's H-function. Namely then the Mellin transform of the solution only consists of fractions of Euler's gamma function having no sum terms. Detailed information about this can be found in the diploma thesis of the author ([Süd1997], sections 5.1.1 and 6.1, p. 65-66 and 75-79).

In Mellin space it is easy to convert by combinatorics and clever cancelling a yielded main solution by use of the reflection formula ([EMOT1953], eqs. 1.2(6) and 1.2(7), p. 3) into further main solutions (cmp. [Mes1959], rem. at p. 41) due to the sampling theorem ([Mar1986], chap. 6, p. 127-131). This possibility is dealed with by Mellin by the name "proper substitution" ([Mel1910], end of §5, p. 314).

Within the setting of this elaboration fortunately it has not been possible to try out the efficiency of the difference equations. In the diploma thesis of S. Hoffmann [Hoff2000] there are several realizations of solution techniques and a lot of interesting references to this topic.

■ 2.4.1.3. Fox's H-functions

The inverse Mellin transformation of fractions of Euler's gamma functions having no sum terms according to Dixon/Ferrar [DF1936] yields a class of functions, which today [MS1978] is called Fox's H-function, because C. Fox [Fox1961] did not know about the elaboration of his predecessors.

Happen fractions of Euler's gamma functions having sum terms to be a solution, then Baumann's ℵ-functions [SBN1998] result, a generalization of Saxena's I-function [Sax1982].

The presentation of Fox's H-function to be an inverse Mellin transform results ([SBN1998], eq. (1), p. 401) for positive $\boldsymbol{A_j}$ and $\boldsymbol{B_j}$ with $\{\boldsymbol{m, n, p, q}\} \in \mathbb{N}_0$ in:

$$\mathcal{H}_{p,q}^{m,n}\left[x \middle| \begin{array}{ll} \{\{a_1, A_1\}, \ldots, \{a_n, A_n\}\} \mid & \{\{a_{n+1}, A_{n+1}\}, \ldots, \{a_p, A_p\}\} \\ \{\{b_1, B_1\}, \ldots, \{b_m, B_m\}\} \mid & \{\{b_{m+1}, B_{m+1}\}, \ldots, \{b_q, B_q\}\} \end{array}\right] = \\ = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\left(\prod_{j=1}^{m} \Gamma[b_j + B_j z]\right)\left(\prod_{j=1}^{n} \Gamma[1 - a_j - A_j z]\right)}{\left(\prod_{j=m+1}^{q} \Gamma[1 - b_j - B_j z]\right)\left(\prod_{j=n+1}^{p} \Gamma[a_j + A_j z]\right)} x^{-z} \, dz . \tag{2.29}$$

Even more complicated Mellin Barnes integrals result, if Riemann's or even Hurwitz's zeta function or better the yet more general Lerch's zeta or phi function occurs in Mellin space to be a solution (examples for this: [Obe1974], formula I.3.24-I.3.26 and A.16, p. 28 and 272). The numerical series of this kind of functions usually is won via the residual theorem by use of the preparatory elaborations of Mellin [Mel1910] and Barnes [Bar1908].

The essential difference between a power series and an analytical function due to Barnes [Bar1908] consists in the fact, that a function has got an analytical continuation possibly to the whole complex number area and corresponding analytical properties, while the power series is interesting for numerics.

In the case of Bessel's differential equation being mentioned as an example, the Mellin transformation yields an easy difference equation, where the inverse Mellin transformation of altogether four main solutions yields not only an optimized fundamental system of two modified Bessel functions of first kind, but also the modified Bessel function of second kind being very important for the propagators. The fourth main solution cannot be subjected to the inverse Mellin transformation.

■ 2.4.2. Momenta Calculation from Laplace Transforms

■ 2.4.2.1. Motivation

Since in this elaboration rather the discussion of the theoretical variance in comparison to measured data is dealed with, it is important to calculate the momenta of a solution function reliably also for the case, that the solution function itself already is too complicated, namely exceeds the setting of Fox's H-functions.

Since momenta according to equation (2.18) are related to Mellin transformation, their calculation principally is easier than the calculation of a completed solution function. The deeper reason for this is the convolution theorem of Mellin transformation ([Obe1974], formulae I.1.13-I.1.15, p. 12), which e.g. converts the Mellin transform of a Fourier convolution into a Fox's H-function, yet.

■ 2.4.2.2. Calculation

Since the Laplce transform of the solution function already has got the spatial coordinates, the momenta of the Laplace transform also can be calculated directly. From the corresponding result the inverse Laplace transform into the time coordinate is to be build.

According to Oberhettinger ([Obe1974], eq. (c'), p. 3) a Mellin transform of a Laplace transformation results to be

$$\mathcal{M}_p^y[\mathcal{L}_t^p[f[x,t]]] = \Gamma[y]\,\mathcal{M}_t^{1-y}[f[x,t]]\,, \tag{2.30}$$

which also enables the calculation of the inverse Laplace transformation via Mellin transformation.

If e.g. the momenta calculation of a Laplace transform yields a Fox's H-function, then because of the result (2.30) also the moment of the distribution results to be a Fox's H-function. The distribution function itself is much more complicated in this case.

■ 2.4.3. Momenta Calculation from Fourier Transforms

■ 2.4.3.1. Direct Calculation

If the Fourier transform of a distribution function is known only, then the momenta calculation turns out to be much more tricky than with the Laplace transform, because now the spatial coordinate itself is not known at all.

In spite of this for symmetrical Fourier transforms (which are to be discussed preferably with driftless diffusion) generally it is possible to derive the following relation, indeed via Mellin transformation of an inverse Fourier transform by use of the Fubini theorem ([SKM1993], eq. (1.32), p. 9), which arranges the swapping of integrals, where again as in relation (2.27) the Mellin transform of the cosine function occurs:

$$< (|x|^m), (\mathcal{F}^{-1})_k^x[f[|k|, t]] > = \int_{-\infty}^{\infty} \frac{|x|^m}{2\pi} \left(\int_{-\infty}^{\infty} f[|k|, t]\, e^{-ikx}\, dk\right) dx =$$

$$= \frac{4}{2\pi} \int_0^{\infty} \left(\int_0^{\infty} \text{Cos}[|k|\ x]\, x^m\, dx\right) f[k, t]\, dk = \tag{2.31}$$

$$= \frac{2^{1+m}\, \Gamma[\frac{1}{2} + \frac{m}{2}]\, \mathcal{M}_k^{-m}[f[k, t]]}{\sqrt{\pi}\ \Gamma[-\frac{m}{2}]} \ .$$

The Mellin transform of the Fourier transform must be very suitable, that the result (2.31) for $\boldsymbol{m \to 2}$ allows the calculation of variance.

■ 2.4.3.2. Momenta of a Fourier Convolution

The momenta of a Fourier convolution result to a fundamentally easier theorem (appendix A of this elaboration), which here at least shall be mentioned:

$$< (x^m),\ f[x] * g[x] > = \sum_{k=0}^{m} \binom{m}{k} < x^k,\ f[x] > \ < x^{m-k}, g[x] > \ . \tag{2.32}$$

The difference between the equations (2.31) and (2.32) mainly consists in the fact, that on the one hand the symmetrical momenta and on the other hand the factual momenta are calculated.

■ 2.5. Summary

Dirac's delta function being important to the variance theorem (1.14) very much has been introduced as an analytical function of the complex number area being consistent due to function theory.

The use of the delta function when applying Laplace and Fourier transformations to a fractional linear equation has been dealed with. The results are optimized fundamental systems of propagators, which eventually already contain the Green's function of the inhomogenity.

The Riesz operator due to B. J. West [SWe1982] being useful for the discussion of driftless diffusion has been introduced as an elegantly arranged Fourier convolution.

The direct Mellin transformation of a linear differential equation often leads to a difference equation, whose solution manifold can widely exceed beyond the Mellin transforms of Fox's H-functions.

The calculation of momenta from Laplace or Fourier transforms of a distribution function has been introduced. This deal simplifies the analytical discussion enormously, since the variance of a solution propagator principally belongs to an easier function class than the solution propagator itself.