Mechanics.nb

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■ 1.1. Question

How can a mechanical problem be modelled by use of *Mathematica*?

■ 1.2. Answer

▪ 1.2.1. The Principle of Least Compulsion According to C. F. Gauß

▪ 1.2.1.1. In Words

The *principle of least compulsion* according to C. F. Gauß [Gauß1829] claims, that the turnover of energy within any dynamics of physics and chemistry always is minimized.

▪ 1.2.1.2. Derivability

This integral principle easily can be described in words, but it cannot be derived conclusively. In any case it is a unification of the principle of Le Chatelier, the principle of Fermat and the integral principle of Hamilton-Jacobi in quite illustrative and consistent way.

▪ 1.2.1.3. Describability

The mathematical way to describe such an integral principle is done by variation calculus. The result of variation calculus are dynamical equations which describe the given dynamical problem in consideration of the integral principle. The equations can be solved by analytical or numerical methods.

▪ 1.2.1.4. Motivation to Use Integral Principles

If the dynamical theory is build up consistently, it is possible to conclude from a balance of (mechanical) forces or a balance of power to a generalized dynamical equation.

Variation calculus is a means of simplification and systematisation when setting up dynamical equations of motion. A limitation to "pure" mechanical problems in this case is rather artificial, but possible.

"Mechanics" in the original sense of Newton means "dynamics" or "description of motion".

■ 1.2.2. Mathematical Description

■ 1.2.2.1. Power (Density) as a Generalized Density of Energy

In order to be able to carry out variation calculus, the balance of energy ***E*** should be interpreted as an integral over the four coordinates of physics (three spatial coordinates and one time coordinate). An integral over time is to be used always when a time-dependent dynamical description is asked for.

The corresponding *"density of energy"* is described by the following expression:

$$\int\int\int\int \mathcal{P}[x,\ y,\ z,\ t]\, dt\, dz\, dy\, dx = E \stackrel{!}{=} \text{Minimum.} \tag{1.1}$$

Even in the absence of the spatial integrations we deal with a balance of power in the integrand of the time integral.

Provided that the four coordinates are independent from each other, the power density is given by partial derivatives of energy with respect to the four coordinates:

$$\mathcal{P}[x,\ y,\ z,\ t] = \frac{d^4 E}{dx\, dy\, dz\, dt} \stackrel{\text{here}}{=} \frac{\partial^4 E}{\partial x\, \partial y\, \partial z\, \partial t} \tag{1.2}$$

Within mechanics it is possible to yield a balance of power by vectorial multiplication of a force balance with the velocities $\vec{\boldsymbol{v}}[\boldsymbol{x},\ \boldsymbol{y},\ \boldsymbol{z},\ \boldsymbol{t}]$. This is also true for friction forces. If the heat production due to friction is also taken into account in the balance of power, then thermodynamics is involved in this case. In any problem of field physics, it is important that terms of the kind $\boldsymbol{x}[\boldsymbol{x},\ \boldsymbol{y},\ \boldsymbol{z},\ \boldsymbol{t}]$ do not emerge. Therefore a proper distinguishing of content and form between coordinates and field variables (within these coordinates) is essential.

The mathematical description of the system is complete if ***n*** field variables obey ***n*** equations.

The statement that the turnover of energy is always (a time-dependent statement!) minimal, yields the result that a variation over the balance of power (density) is the balance of power (density) of all external forces controlling the system. If there are no controlling forces, the result of variation is zero.

With this ansatz, at least, a lot of linear dynamical equations can be understood. It is possible to go into detail if needed.

▪ 1.2.2.2. Theoretical Special Case of Systems Without Power

Especially in "classical mechanics" there is a custom to neglect friction etc. This dynamics in most cases is described via the integral principle of Hamilton-Jacobi which is included in the *principle of least compulsion* according to Gauß:

In order to get a balance of power which is not equal to zero, a difference quotient of the balance of energy over the time is used in Hamiltonian potential systems, instead of its derivative with respect to time (1.2), whereas the time integral in equation (1.1) remains valid.

Another wellknown convention is to assume that the total energy of the system $E[t]$ at time t_1 consists of potential energy $E[t_1] = V$, and at time t_2 of kinetic energy $E[t_2] = T$, respectively.

The existence of just two balances of energy at two different times is enough to apply variation calculus. If the solution of the dynamical equation yields that the potential energy reaches zero only when $t_2 \to \infty$, another way of calculation should be searched for to avoid this problem.

The result of these considerations can be found (partially with completely different derivations) under the keyword *variation principle of Hamilton-Jacobi* in the literature (e.g. [Bau1954], V, §14, pages 42-46):

$$\int_{t_1}^{t_2} \frac{E[t_2] - E[t_1]}{t_2 - t_1}\, dt = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (T - V)\, dt \stackrel{!}{=} \text{Minimum} \tag{1.3}$$

In recent publications, the time-dependent denominator in equation (1.3) was omitted (so [Bau1954], page 44) which does not change the result of variation calculus as long as external controlling forces are missing. In German, the time integral over the balance of energy is called *"Wirkung" = "effect"*, where is overlooked that energy itself already is an *"integral of motion"* thus being an effect of dynamics. In English, the derivative of energy with respect to time also is called (power) effect.

Those who believe to have understood the variation principle of Hamilton-Jacobi, should extend it to systems having a real (power) effect, in order to be able to discuss the variants of the principle of least compulsion.

▪ 1.2.2.3. Ostrogradski Equation

To perform variation derivative in physical problems correctly, the following formula with $1 \le i \le N$ and $i \in \mathbb{N}$ is needed:

$$\frac{\delta F}{\delta q_i} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \left(\frac{-d}{dx}\right)^{n_1} \left(\frac{-d}{dy}\right)^{n_2} \left(\frac{-d}{dz}\right)^{n_3} \left(\frac{-d}{dt}\right)^{n_4} \frac{\partial F}{\partial q_i^{(n_1,n_2,n_3,n_4)}[x, y, z, t]} \tag{1.4}$$

This result is due to the hint of Smirnov ([Smi1988], equation (36) and (37), page 183) and is called *Ostrogradski equation*. The functional F depends on corresponding field variables q_i, whereby the counting index $1 \le i \le N$ allows N different field variables.

As a rule, the infinite series in formula (1.4) are finite, respectively. The corresponding *Mathematica* modules for the calculation of functional derivatives have already been successfully presented by Professor Dr. Gerd Baumann in his lectures on "*Mathematica* in Theoretical Physics" (WS1995-SS1996 University of Ulm).

To understand the Ostrogradski equation (1.4) compared to the hint of Courant/Hilbert [?], one needs an example explaining the transition from functionals for which Schwarz's sentence ($\partial_{x,y} f[x, x] = \partial_{y,x} f[x, y]$) is not applicable to those for which it is the case.

In physics, Ostrogradski formula (1.4) leads from a balance of (power) effect $F \to \mathcal{P}$ being properly set up in N field variables to N different consistent dynamical equations.

■ 1.2.3. Outlook

By use of the *principle of least compulsion* in the way shown above (which is extendable) a lot of dynamical systems in physics and technology can be described consistently in terms of (power) effect.

Setting up dynamical equations can be automated by use of *Mathematica*. If discrepancy occurs between the theory shown here and other derivations of dynamical equations, the localization and correction of the mistakes is to be aimed at. In this way, a collegial teamwork is possible also in a field of mathematical physics which is inscrutable at once.

Setting up the balance of forces or power can be mastered also by technical engineers whose knowledge of mathematics is rather scarce. The additivity of power or forces can be used.

Literature

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